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1973 J. Phys. A: Math. Nucl. Gen. 6 L79

(http://iopscience.iop.org/0301-0015/6/7/002)

J. Phys. A: Math., Nucl. Gen., Vol. 6, July 1973. Printed in Great Britain. © 1973

LETTER TO THE EDITOR

Coordinate free canonical commutation relations

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Received 7 May 1973

Abstract. We present an algebraic argument which shows that the coordinate free canonical commutation relations of Segal are the natural extension of the coordinate–velocity commutation relations. We do not invoke the Schrödinger representation. We define an algebra of quantum observables for any manifold and show it becomes the usual algebra when the manifold is riemannian.

The canonical equal-time commutation relations for a quantum-mechanical system whose configuration space is a riemannian manifold M with metric g are (Bloore and Routh 1973)

$$[q^i, \dot{q}^j] = \mathrm{i}g^{ij}(q), \qquad i,j = 1, \ldots, n, \tag{1}$$

$$[\dot{q}^i, \dot{q}^j] = \tfrac{1}{2}\mathrm{i}\{g^{ia}\Gamma^j_{ak} - g^{ja}\Gamma^i_{ak}, \dot{q}^k\} + \mathrm{i}K^{ij}(q), \tag{2}$$

and

$$[q^i, q^j] = 0.$$

Square and curly brackets denote commutator and anticommutator, Γ is the Christoffel symbol and K is some antisymmetric tensor function of q.

It is aesthetically desirable (and actually necessary for problems such as the action–angle uncertainty relation where the manifold cannot be covered by a single coordinate patch), to express these equations in a form which is independent of choice of a coordinate system. That is the function of this letter.

It follows from equation (1) that the time-derivative $\dot{\phi}$ of any function $\phi(q)$ is

$$\dot{\phi}(q, \dot{q}) = \tfrac{1}{2}\{\phi_{,i}, \dot{q}^i\}, \tag{3}$$

where

$$\phi_{,i} = \partial\phi/\partial q^i.$$

To prove equation (3), we observe that both sides are derivations on the function ϕ and that equation (3) holds for the functions $\phi(q) = q^j$. Two derivations which agree on the basic elements q^j will agree on all functions of the q^j. If new coordinates $q^{1'}, \ldots, q^{n'}$ are functions of $q^1, \ldots, q^n$, then by setting $\phi(q) = q^{i'}(q)$ in equation (3) we obtain

$$\dot{q}^{i'} = \tfrac{1}{2}\{q^{i'}{}_{,i}, \dot{q}^i\}. \tag{4}$$

Thus

$$\{\phi_{,i'}, \dot{q}^{i'}\} = \{\phi_{,i'}, \tfrac{1}{2}\{q^{i'}{}_{,i}, \dot{q}^i\}\} = \{\phi_{,i}, \dot{q}^i\} \tag{5}$$

so that equation (3) holds in all coordinate systems. It follows that the tensor form of equations (1) and (2) is preserved in all coordinate systems also.

The crucial observation is that equation (5) extends to the case when $\phi_{,i}$ is replaced by the covector field X_i associated with an arbitrary vector field X on M:

$$\tfrac{1}{2}\{X_i, \dot{q}^i\} = \tfrac{1}{2}\{X_{i'}, \dot{q}^{i'}\} \equiv V^*(X). \tag{6}$$

We shall use g to change the elevation of suffixes. Evidently $V^*(X)$ is linear in X so we may regard V^* as a kind of cotangent field, that is a momentum. We note finally that equation (1) implies that (Bloore and Routh 1973)

$$[\phi, \dot{q}^i] = \mathrm{i}g^{ij}\phi_{,j},$$

which in turn gives the coordinate free result

$$[\phi, V^*(X)] = \mathrm{i}X^j\phi_{,j} = \mathrm{i}X\phi, \tag{7}$$

where $X\phi$ is the function on M resulting from the action (Hicks 1965) of the vector field X on the scalar field ϕ. The equations (6) and (7) suggest the following formulation of an algebra of observables for a quantum-mechanical system with classical configuration manifold M.

Let M be a real C^∞ n-manifold and let $\mathscr{F}$ be the associative abelian algebra of complex-valued C^∞ functions on M. Let $\mathscr{X}$ be the Lie algebra of C^∞ vector fields on M. Without yet furnishing M with a metric, we may enlarge the algebra $\mathscr{F}$ to a Lie algebra by adjoining to it formal elements $V^*(X)$ for all $X \in \mathscr{X}$.

Let $\mathscr{B}$ be the Lie algebra over the complex numbers of formal polynomials in ϕ and $V^*(X)$ for all $\phi \in \mathscr{F}$, $X \in \mathscr{X}$, with the commutation relation

$$[\phi, V^*(X)] = \mathrm{i}X\phi. \tag{8}$$

We postulate also that V^* is linear,

$$V^*(X+Y) = V^*(X) + V^*(Y), \qquad V^*(cX) = cV^*(X).$$

We seek a commutation relation of the form

$$[V^*(X), V^*(Y)] = \mathrm{i}V^*(Z) + \mathrm{i}K(X, Y) \tag{9}$$

where K is some antisymmetric tensor field of type (0, 2) on M. For brevity define the Jacobi function

$$J(x, y, z) = [x,[y, z]] + [y,[z, x]] + [z,[x, y]].$$

The equations (8) and (9) will be consistent if and only if

$$J(\phi, V^*(X), V^*(Y)) = 0, \tag{10}$$

$$J(V^*(X), V^*(Y), V^*(Z)) = 0. \tag{11}$$

Equation (10) requires that Z is the Lie product (Hicks 1965)

$$Z = -[X, Y] \tag{12}$$

and equation (11) further requires that K is a closed two-form,

$$K(X,[Y,Z]) + XK(Y,Z) + \text{cyclic terms} = 0. \tag{13}$$

Equations (8), (9), (12) and (13) were obtained by Segal (1960) from consideration of the Schrödinger representation of the Heisenberg position–momentum commutation relations.

We denote by $\mathscr{A}$ the algebra $\mathscr{B}$ with the further identification (9) with Z given by (12) and K obeying (13). We call this algebra the algebra of quantum observables for the manifold M. We make the usual physical interpretation that a state ω of the system is a normed positive linear functional on $\mathscr{A}$, that $\omega(\phi)$ is the expectation value of the observable ϕ in the state ω, that $\omega(V^*(X))$ is the expectation value of the observable $V^*(X)$, where $V^*(X)$ is the scalar field which results from the action of the momentum covector V^* on the element X of the tangent bundle.

The commutation relations (8) and (9) with (12) and (13) do not relate $V^*(X)$ and $V^*(\phi X)$ where ϕX is the vector field X scaled by the multiplier $\phi \in \mathscr{F}$. They are consistent with the equation

$$V^*(\phi X) = a\phi V^*(X) + (1-a) V^*(X)\phi$$

with any choice of the constant a. Let us now suppose the manifold M is riemannian with metric tensor g, and to each $\phi \in \mathscr{F}$ define grad $\phi \in \mathscr{X}$ by

$$g(\text{grad } \phi, X) = X\phi$$

and define $\dot{\phi} \in \mathscr{A}$ by

$$\dot{\phi} = V^*(\text{grad } \phi).$$

It is then easy to verify that the dot operation is a derivation from $\mathscr{F}$ into $\mathscr{A}$, that is,

$$(\phi\psi)^{\cdot} = \dot{\phi}\psi + \phi\dot{\psi}$$

if and only if

$$a = \tfrac{1}{2}.$$

Henceforth let us assume this, so that

$$V^*(\phi X) = \tfrac{1}{2}\{\phi, V^*(X)\}. \tag{14}$$

We may now make the connection with the coordinate version of the commutation relations (1) and (2). We suppose that the riemannian manifold M permits a global coordinate system $q^1, \ldots q^n$. Then $q^i \in \mathscr{F}$ and $\dot{q}^i = V^*$ (grad q^i). Equation (8) then gives

$$[q^i, \dot{q}^j] = \mathrm{i}g\,(\text{grad } q^i, \text{grad } q^j) = \mathrm{i}g^{ij}$$

which is equation (1), and equation (9) gives

$$\begin{aligned}[\dot{q}^i, \dot{q}^j] &= -\mathrm{i}\, V^*([\text{grad } q^i, \text{grad } q^j]) + \mathrm{i}\, K(\text{grad } q^i, \text{grad } q^j) \\ &= \mathrm{i}\, V^*((g^{ia}\Gamma^j_{ak} - g^{ja}\Gamma^i_{ak})\text{grad } q^k) + \mathrm{i}\, K^{ij}\end{aligned} \tag{15}$$

which by equation (14) reduces to equation (2).

We thank Mr L Routh for helpful discussions.

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